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Series Expansion and CAM Study of the Nonuniversal Behavior of the Symmetric 16-Vertex Model

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The phase diagram and the critical indices are investigated for the symmetric 16-vertex model on the square lattice by combining a variational series expansion and the coherent anomaly method. The nonuniversal critical exponents smoothly interpolate between two exactly solvable cases, namely the Baxter eight-vertex model and the Ising model.

KEY WORDS: Vertex model; critical behavior; nonuniversality; coherent anomaly method; series expansion.

1. INTRODUCTION

Critical phenomena are still one of the most exciting topics of statistical physics. In particular, universality breaking is of great importance. The universality hypothesis[1] states that the critical indices do not depend on the details of a specific Hamiltonian. In other words, the critical exponents are constant for a given universality class of models. This hypothesis seems to be generally valid apart from few exceptions. Historically the first violation of the universality was observed in the Baxter eight-vertex model (B8V),[2] the critical indices of which turned out to be continuous functions of the model parameters. The second nonuniversal model, the Ashkin–Teller (AT)[3,4] model, is in fact related to the B8V. To our knowledge the last exception is represented by the antiferromagnet next-nearest-neighbor Ising model (ANNNI).[5–10] This model cannot be solved exactly, but there exists strong evidence for nonuniversal behavior originating in approxi-

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mate studies such as the perturbation expansion,[5,8] MC renormalization group,[6] MC simulations,[7,9] or, quite recently, the coherent anomaly method (CAM).[10,11] In connection with universality breaking, Suzuki[12] proposed a new concept, so-called weak universality, which is based on the fact that instead of the temperature difference, the most natural measure for the distance from the critical point is the inverse correlation length. He suggested that the genuine universal indices are the renormalized exponents $(2-\alpha)/\nu$, β/ν, γ/ν, δ, etc. While the B8V and the AT models are weakly universal, there are numerical indications that even weak universality is violated near the frustration point of the ANNNI model. Unfortunately, the approximate methods fail in the vicinity of this peculiar point and, consequently, there is no definite solution of this interesting problem.[10]

The main goal of the present work is to reveal a new nonuniversal subspace in the general 16-vertex model on the square lattice, namely the symmetric 16-vertex model (S16V).[13] We intend to describe its phase diagram and to calculate several critical exponents along the critical manifold.

Before describing our approach, let us recall the known results on the S16V model and introduce the notation. The model lives on the square lattice of Λ sites and with cyclic boundary conditions. To each edge of this lattice we attribute a two-state variable ("spin") so that the edge can be either in the state $(+)$ or in the state $(-)$. With each node we associate an energy according to the vertex configuration of incident edge states. These vertex energies are considered to be independent of the permutation of adjacent edge states. That is why we can use the notation ε_i and $a_i = \exp(-t\varepsilon_i)$ for the vertex energy and the Boltzmann weight of a vertex with exactly i incident edges in the state $(-)$ and the remaining $(4-i)$ edges in the state $(+)$. Hereafter t stands for the inverse temperature. The statistical weight of a lattice configuration of edge states is the product of vertex weights of all vertices, and the partition function is given as the sum running over all possible configurations of edge states:

$$Z(\{a_i\}, \Lambda) = \sum \prod (\text{weights}) \tag{1a}$$

Our task is to calculate the dimensionless free energy per site f defined by

$$-f(\{a_i\}) = \lim_{\Lambda \to \infty} \frac{1}{\Lambda} \log Z(\{a_i\}, \Lambda) \tag{1b}$$

and to grasp its nonanalytical properties.

The described model is of great interest, partially because it encompasses (in its special cases) other outstanding models, such as the Ising and

the eight-vertex models in a field,[14] and is closely related to polymer systems or lattice animals. However, not very much is known about it. Most of the results have been derived from its transformation properties. It is well known that the general vertex models are gauge-invariant.[15,16] In the present case there exists a one-parameter $O(2)$ gauge transformation of the vertex weights preserving the permutation symmetry of the model[13]:

$$\begin{pmatrix} \tilde{a}_0 \\ \tilde{a}_1 \\ \tilde{a}_2 \\ \tilde{a}_3 \\ \tilde{a}_4 \end{pmatrix} = \frac{1}{(1+y^2)^2} \begin{pmatrix} 1 & 4y & 6y^2 & 4y^3 & y^4 \\ y & 3y^2-1 & 3y^3-3y & y^4-3y^2 & -y^3 \\ y^2 & 2y^3-2y & y^4-4y^2+1 & 2y-2y^3 & y^2 \\ y^3 & y^4-3y^2 & 3y-3y^3 & 3y^2-1 & -y \\ y^4 & -4y^3 & 6y^2 & -4y & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \tag{2}$$

The partition function of the transformed system characterized by the weights $\{\tilde{a}_i\}$ is exactly equal to the statistical sum of the original model, $Z(\{\tilde{a}_i\}, \Lambda) = Z(\{a_i\}, \Lambda)$. In what follows, the gauge parameter y is taken as real-valued.

As a consequence of gauge invariance there is a redundancy in the vertex weights of the model. Specifically, it is always possible to perform a gauge transformation resulting in equality between the weights a_0 and a_4. Thus, it is sufficient to investigate only models with this constraint, without any loss of generality: having the complete solution in this higher-symmetric subspace, one can recover any quantity concerning the original model by using the inverse gauge transformation.

As shown by Wu and Wu,[13] in the vertex-weight space there exists a self-dual manifold

$$a_0^2 a_3 - a_1 a_4^2 - 3a_2(a_0 - a_4)(a_1 + a_3) + (a_1 - a_3)[a_0 a_4 + 2(a_1 + a_3)^2] = 0 \tag{3}$$

which is, of course, an invariant of the transformation (2). The self-dual property of (3) consists in the fact that each point on this hypersurface can be mapped into itself by choosing an appropriate (point-dependent) value of gauge parameter y. It has been observed[13] that in the ferromagnetic Ising subspace of the S16V, (3) coincides with the critical locus $H = 0$. Clearly hypersurface (3) is a good candidate for the location of phase transitions. Actually, according to our numerical investigation described below, the self-dual manifold supports all the phase transition points. The critical points divide this manifold into two parts: one of them is the first-order transition locus and the other corresponds to the disordered phase. It is possible, in principle, that there exists another manifold of phase

transition points in the space of vertex weights of the symmetric model, but so far we have not located it.

There are two exactly solvable subspaces in the S16V model, providing a check on our calculations. The first subspace is given by $a_0 = a_4$, $a_1 = a_3 = 0$, which is in fact the special case of the B8V (with $b = c = d$), and the second one, being the intersection of hypersurfaces (3) and $a_0 a_2 a_4 - a_0 a_3^2 - a_1^2 a_4 + 2a_1 a_2 a_3 - a_2^3 = 0$, corresponds to the zero-field Ising model.[14] In both cases there exists a critical point, and it is worth noting that these critical points belong to different universality classes. In particular, the exponent ν is equal to 1.5 for the mentioned B8V case,[2] while $\nu = 1$ for the Ising-like critical point. Because of the local character of the mapping between S16V and the Ising model[14] the critical exponent $\nu = 1$ is expected to be shared by the S16V, too. Now we have to answer the question of whether or not these two critical points lie on the same critical curve. If this is the case, we have a strong indication for nonuniversal behavior. This is the main motivation for the present investigation.

The rest of the paper proceeds as follows. In the next section we sketch our method, which combines a series expansion technique and a variational method (with the free gauge parameter taken as the variational one), resulting in a series of classical approximations for the S16V. This method was first introduced and tested in ref. 17 on the Ising antiferromagnet. Section 3 reports our results concerning the phase diagram, and the critical indices related to the polarization and the polarization susceptibility are calculated within the CAM in Section 4. In Appendix A we describe the "no-free-ends" method for generating the series expansion and give some useful data. Appendix B is devoted to the analytical explanation of the variational series expansion method in its lowest approximation order.

2. VARIATIONAL SERIES EXPANSION METHOD

The property of gauge invariance allows us to treat the model in its transformed picture with the vertex weights $\tilde{a}_i$ which are functions of the original weights a_i and the free gauge parameter y. We can construct the following *formal* series expansion for the free energy of the model:

$$-\mathscr{F}_L(\{a_i\}, y) = \log(\tilde{a}_0) + \sum_{n=2}^{L} \mathscr{F}^{(n)}(\{\tilde{a}_i/\tilde{a}_0\}_{i=1}^{4}) \tag{4}$$

Here, L denotes the approximation order and $\{a_i\}$ is the complete list of vertex weights. In this expansion, the lattice configuration with all edges in the state $(+)$ is supposed to be the ground state in the transformed picture and gives the logarithmic contribution on the RHS of (4). The successive

excitations are generated from the sea of (+) edge states by changing the (+) edge signs into the (−) ones on a finite number of edges. The $\mathscr{F}^{(n)}$ terms are homogeneous polynomials of the order n,

$$\mathscr{F}^{(n)}(\{\tilde{a}_i/\tilde{a}_0\}_{i=1}^4) = \sum_{\substack{n_1,n_2,n_3,n_4 \\ n_1+n_2+n_3+n_4=n}} c(n_1, n_2, n_3, n_4) \times \left(\frac{\tilde{a}_1}{\tilde{a}_0}\right)^{n_1} \left(\frac{\tilde{a}_2}{\tilde{a}_0}\right)^{n_2} \left(\frac{\tilde{a}_3}{\tilde{a}_0}\right)^{n_3} \left(\frac{\tilde{a}_4}{\tilde{a}_0}\right)^{n_4} \tag{4'}$$

and represent the contributions of all graphs with just n nodes with at least one incident edge in the (−) state. Details concerning the series calculation and the "no-free-ends" data which enable the reader to recover our expansion up to the $L=15$ order are reported in Appendix A.

We emphasize that $\tilde{a}_i$ in (4) depend on the given vertex weights a_i *and* on the gauge parameter y. Obviously, if the limit $\lim_{L\to\infty} \mathscr{F}_L$ exists, then it does not depend on y. On the other hand, the way in which the series approaches its limiting value certainly *does*. This is the main point of our method. We try to choose y in such a way that it makes the series properly convergent. Actually, it turns out that there exists a region of moderate values of y in which $\mathscr{F}_L$ as a function of y exhibits a plateau (for a fixed set of original weights) while outside of this interval it oscillates with a growing amplitude, indicating the loss of the series convergence. In order to restore at least the "local independence" on the gauge parameter, we impose the stationarity condition

$$\frac{\partial}{\partial y} \mathscr{F}_L(\{a_i\}, y) = 0 \tag{5}$$

which is taken as an implicit definition of y as a function of the original statistical weights. Naturally, this equation can possess more than one solution in the above-mentioned interval of well-behaved y values. Let us denote by $S=\{y_i\}$ the set of those solutions to (5) for which the series (4) converges. Among the members of S we choose the solution $\bar{y}$ which implies the minimal value for the free energy approximation f_L:

$$f_L(\{a_i\}) = \mathscr{F}_L(\{a_i\}, \bar{y}_L) = \min_{y_i \in S} \mathscr{F}_L(\{a_i\}, y_i) \tag{6}$$

The described procedure [i.e., the second equality in (6) together with the implicit definition (5) and the convergence restriction] defines a unique solution to (5) as a function of the original vertex weights: $\bar{y}_L = \bar{y}_L(\{a_i\})$, with L denoting the dependence on the approximation order. We will call $\bar{y}_L$ the physical solution to the stationarity condition.

As a self-consistent approach, the obtained variational approximation f_L represents for fixed L a kind of the mean-field-type theory. The critical behavior can be explicitly calculated [see formulas (18) and (23)–(24) in Section 4] and turns out to be governed by the singularities of the function $\bar{y}_L$. For example, the first-order phase transition takes place when the role of the physical solution to the stationarity condition is transferred from one root y_1 to the other y_2 such that $y_2 \neq y_1$. In such a case $\bar{y}$ exhibits a jump $y_1 \to y_2$ at the transition point, and gives rise to the discontinuity of the first derivative of the free energy. Now one can comprehend the reason for which the first-order transitions are located on the self-dual manifold (3). Let us suppose that the system passes through this manifold when its temperature varies. The self-dual property ensures the existence of two gauge parameters y_1 and y_2 for which the transformation (2) gives the statistically equivalent weights $\{\tilde{a}_i(y_1)\} = \{(-1)^i \tilde{a}_i(y_2)\}$ (the partition function is invariant under the simultaneous negation of a_1 and a_3). Let us call the couple $\{y_1, y_2\}$ the self-dual roots. As a consequence, $\mathscr{F}_L(\{a_i\}, y_1) = \mathscr{F}_L(\{a_i\}, y_2)$ and *if, moreover,* this value is the minimum with respect to S, then $\bar{y} = y_1$ approaching the self-dual manifold from one side while $\bar{y} = y_2$ approaching the self-dual manifold from the other side. This means that $\bar{y}$ is discontinuous, which corresponds to the first-order phase transition. Obviously, there can exist (and in fact exists) a region of the self-dual manifold lacking this discontinuity in the first derivative of the free energy, because the roots y_1 and y_2 do not provide the minimum of $\mathscr{F}_L$.

The situation is a bit more complicated in the vicinity of a critical point where the stationarity condition has a degenerate root. In order to see this, one can derive from (5) a formula for the temperature evolution of the $\bar{y}_L$:

$$\frac{\partial \bar{y}_L}{\partial t} = -\frac{\sum_{i=0}^{4} \partial_t a_i \, \partial_{y a_i} \mathscr{F}_L}{\partial_{yy} \mathscr{F}_L} \tag{7}$$

which tells us that the critical point can be located using the equation

$$\partial_{yy} \mathscr{F}_L(\{a_i^c\}, y)|_{y = \bar{y}_L^c} = 0 \tag{8}$$

Here, $\{a_i^c\}$ stands for the list of critical vertex weights and $\bar{y}_L^c$ represents the critical value of $\bar{y}$. It turns out that the degenerate solution implied by (8) arises as the result of the coalescence of two above-mentioned self-dual roots, i.e., when y_1 tends to y_2.

To get a deeper insight, consider a model which is self-dual throughout the entire temperature range or, in other words, a model whose temperature evolution is restricted to the self-dual manifold (3) (there exist two classes of such systems, namely the "spin-flip" symmetric model with $a_0 = a_4$,

$a_1 = a_3$ and the B8V in a field with $a_1 = a_3 = 0$). Then, the low-temperature phase corresponds to the region where the self-dual roots, both being real numbers, provide a minimum to $\mathscr{F}_L$ and, consequently, one of them is to be chosen as the physical solution. The choice between these candidates is dictated by the sign of an infinitesimally small external field which disturbs the spin-flip symmetry: y_1 and y_2 give the same free energy, but they differ in the sign of the resulting order parameter—polarization. As the temperature rises, y_1 and y_2 approach one another and, finally, passing the critical point where $y_1 = y_2$, they become complex. The role of the physical solution in the high-temperature phase is played by a new root of (5). Because $\bar{y}$ should be continuous at the criticality, $\bar{y}^c$ is in fact a threefold degenerate solution to (5) and, consequently, besides (8), the third derivative

$$\partial_{yyy} \mathscr{F}_L \mid_{y = \bar{y}_L^c}$$

equals zero, too. The root structure of (5) is the same as above also for a general S16V.

In order to make clearer the interplay among the roots to the stationarity condition (5) in the vicinity of the critical point, we present an exact analytic treatment of the lowest-order approximation in Appendix B.

Before proceeding with our results, it may be useful to give some remarks on the physical interpretation of the described approach and, particularly, to elucidate why the artificially introduced variational parameter y controls the phase transitions. We intend to show that our method is closely related to the effective field approach. Let us imagine that we have to calculate the free energy of the model on the lattice with open boundaries and with an uncorrelated effective field acting on the boundary edge states. In order to determine the effective field, one has to impose some self-consistency condition, resulting in a mean-field-type critical behavior. Now we slightly reformulate the problem in terms of the gauge transformation. The free ends on the lattice boundary can be represented by one-coordinated vertices with an appropriate weight vector $\{w_{ef}(+), w_{ef}(-)\}$, which represents the influence of the imposed effective field (the precise dependence on the field is unimportant). In such a formulation, the model is gauge invariant, and the new vertex weights are transformed under (2) according to

$$\{w_{ef}(+), w_{ef}(-)\} \to \{w_{ef}(+) + y w_{ef}(-),\ y w_{ef}(+) - w_{ef}(-)\}/(1 + y^2)^{1/2}$$

If we fit $y = w_{ef}(-)/w_{ef}(+)$, then the boundary conditions become fixed with all edge states equal $(+)$. Thus, after transformation the role of the effective field is played by the gauge parameter and the fixed boundary

conditions prefer the "gound state" formed by all edges in the state (+), which allows us to construct a formal series expansion for the free energy. For large enough lattice one can neglect the boundary effects, and the size-extensive part of the free energy is given by the series expansion (4). What remains is the specification of the self-consistency condition. Because there are no terms reflecting the existence of the boundary in our expansion, the most natural choice is to require the minimum of the free energy. Then the exact relation between the originally introduced effective field and the present gauge parameter becomes irrelevant and it is sufficient to deal with the latter with the natural restriction to the region (in y) where the series expansion is convergent. This is precisely our formulation of the variational series expansion approach. The advantage over the conventional treatment is that we are able to take into account larger clusters and to extract explicitly the singular parts of the relevant quantities. It is clear that the resulting "physical" value $\bar{y}$ is intimately connected with the order parameter because the latter generates the effective field. This relation is explicitly calculated for the case of the lowest-order approximation in the Appendix B.

3. PHASE DIAGRAM AND THE CRITICAL INDEX ν

We have generated the series expansion (4) up to the 15th order as described in Appendix A. Thus, we are able to calculate the free energy of a general S16V model with a high accuracy. In what follows we restrict ourselves to the mentioned spin-flip symmetric system defined in an appropriate energy scale as

$$\begin{aligned} &\varepsilon_0 = \varepsilon_4 = 0, && a_0 = a_4 = 1 \\ &\varepsilon_1 = \varepsilon_3, && a_1 = a_3 = \exp(-t\varepsilon_1) \\ &\varepsilon_2 = 1, && a_2 = \exp(-t) \end{aligned} \tag{9}$$

As far as the critical properties are concerned, this simplification does not mean a loss of generality: having a given general S16V, we can transform it using (4) to the system with $a_0 = a_4$. Our numerical investigation of such a model showed that, as was expected, the discontinuities and the critical points of the function $\bar{y}$ are strictly located on the self-dual manifold (3), which is, for $a_0 = a_4$, specified by $a_1 = a_3$. Clearly, (9) represents nothing but the parametrization of this self-dual manifold. It should be noted here that due to the spin-flip symmetry, $\bar{y}$ is equal to 1 everywhere in the high-temperature phase for an arbitrary approximation degree L. This

corresponds to the disorder boundary conditions $w_{ef}(+) = w_{ef}(-)$ in the above-described "finite lattice picture."

For a given value of ε_1, we have calculated, using the previously derived formula (8) with $\bar{y}_L^c = 1$, the inverse critical temperature t_L^c in each approximation order L. According to the finite-size scaling,[18] at the critical point t_L^c calculated in the approximation which considers the clusters of sizes $\leqslant L$, the correlation length of the true system is proportional to L. Therefore, near the exact critical point t^c, t_L^c has the asymptotic behavior

$$t_L^c \sim t^c - aL^{-1/\nu} \tag{10}$$

where a depends on the particular series of approximations and on the position on the phase boundary. From (10) one can estimate the inverse critical temperature t^c as well as the critical index ν. However, the straightforward least-squares fit works well only for $\varepsilon_1 \leqslant 1.5$, but for ε_1 large it fails. The reason is obvious. For large ε_1 we obtain $(t^c - t_L^c)/t^c \ll 1$ in each order L, and an arbitrarily small dispersion of data with respect to the ideal power-low dependence (10) causes the failure of the fitting procedure, which then leads to $(t^c, \nu) \to \infty$. In order to overcome this problem, we have fitted the data first by the second-order polynomial $t_L^c = t^c + a/L + b/L^2$. Then we inserted the obtained value of t^c into the large-L asymptotic (10) and used the standard least-squares fit for calculating the index ν. This procedure reproduces the results obtained for $\varepsilon_1 \leqslant 1.5$ by the direct method, and works also for large ε_1.

In Fig. 1 we report some t_L^c data illustrating the convergence of our series of approximations. Figure 1b deserves special attention. There we have chosen

$$\varepsilon_1 = 1 - \log(1 + \sqrt{2})/\log(5 + 4\sqrt{2}) = 0.6275...$$

namely the value for which the exact Ising-like critical point $t_I^c = \log(5 + 4\sqrt{2}) = 2.3662...$ is known.[14] Using the above-described procedure, we arrived at the value $t^c = 2.3458...$, which is consistent with the exact one to within 0.9%. Another test of accuracy is provided by the B8V-like case, whose critical point has components $a_1^c = a_3^c = 0$ $(\varepsilon_1 \to \infty)$, $a_0^c = a_4^c = 3a_2^c$. Here, because of the symmetry of the model, we have reached the exact result in each order L, including the lowest one. Thus, in spite of not too high a maximal order $L = 15$, the method seems to be quite accurate.

Figure 2 presents the phase diagram of the spin-flip-symmetric model. The dashed line represents the critical manifold in the lowest-order

approximation, while the $L \to \infty$ extrapolation is shown by the solid line. The region below this boundary corresponds to the ordered phase with a nonzero spontaneous polarization (which will be discussed in the next section). It is seen from the phase diagram that what was mentioned in the Introduction is actually true: the exactly known critical points, namely the Ising-like point and the one of the B8V model, are connected through one critical line. As is evident from Fig. 3, the critical index ν is a decreasing function of the critical vertex weight a_1^c. For the Ising-like point we have obtained the value $\nu = 0.933...$ and for the B8V case $\nu = 1.413...$, in good

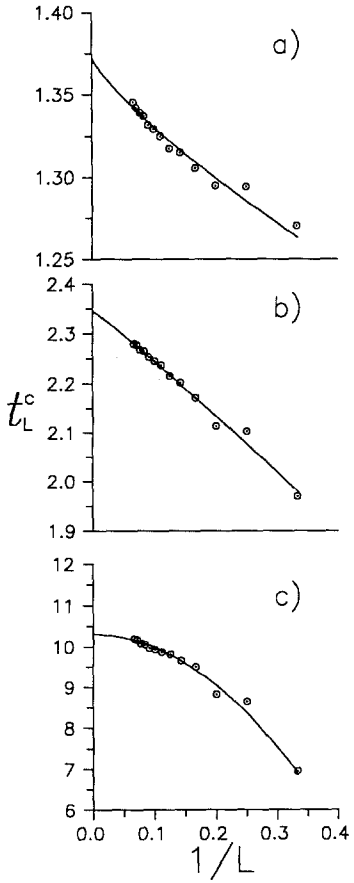

Fig. 1. The dependence of the inverse critical temperature calculated in the approximation order L, t_L^c, on $1/L$ ($L = 3,..., 15$), for three values of the vertex energy ε_1: (a) $\varepsilon_1 = 2$ ($\nu > 1$), (b) $\varepsilon_1 = 0.6275...$ ($\nu = 1$), (c) $\varepsilon_1 = 0.1$ ($\nu < 1$). Data are fitted according to the large-L asymptotics (10) as explained in the text.

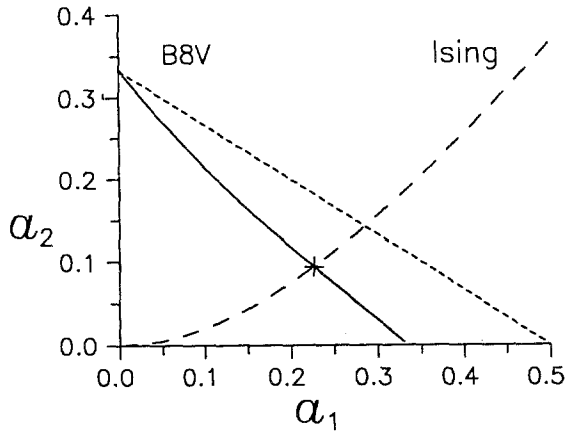

Fig. 2. The phase diagram of the spin-flip S16V model depicted in the (a_1, a_2) plane. The critical lines obtained in the lowest $L=0$ approximation and the $L \to \infty$ extrapolation are represented by the dashed and solid lines, respectively. For illustration, the Ising subspace of the model with the exact critical point denoted by a cross and the B8V critical point are also shown.

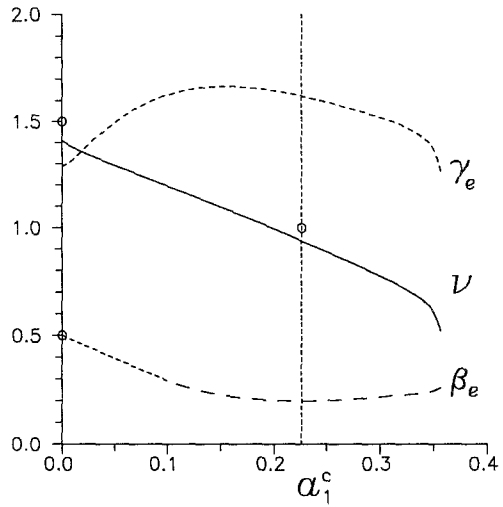

Fig. 3. The results for the critical indices ν, β_e, γ_e as functions of the a_1^c coordinate of the critical point obtained within the CAM. The vertical line corresponds to the Ising-like critical point; the known exact results for the critical indices ν, β_e are represented by circles.

agreement with the exact values $\nu = 1$ and 1.5, respectively. Thus, we arrive at the main result of this work: the symmetric 16-vertex model violates the universality hypothesis.

4. POLARIZATION AND SUSCEPTIBILITY

In this part, we are concerned with the determination of the statistical quantities which are relevant for the description of the critical behavior of the S16V. We first introduce the mean concentrations c_i of the vertices with a given weight a_i:

$$c_i = -a_i \frac{\partial}{\partial a_i} f(\{a_j\}) \qquad (i = 0,..., 4) \tag{11}$$

(c_i are constrained by the normalization condition $\sum_i c_i = 1$.) The concentrations can be viewed as order parameters and allow us to define the generalized susceptibilities via

$$\chi_{ij} = -\frac{\partial}{\partial \varepsilon_j} c_i \tag{12}$$

An arbitrary susceptibility can be expressed in terms of the χ_{ij}, which are easily calculable. Let us now define a directly measurable order parameter, namely the polarization

$$p = \tfrac{1}{4} \sum_i (4 - 2i)\, c_i \tag{13}$$

where the sum runs over all values of $i = 0,..., 4$. Physically, p represents the mean value of the edge-spin variable and its definition coincides with the one in the B8V. Turning on an external field E linearly coupled to the state variables, the vertex weights transforms to $a_i = a_i(E = 0) \exp[tE(4 - 2i)/2]$. Using the evident relation $\partial_{\varepsilon_i} = -t a_i \partial_{a_i}$, we easily derive the formula for the polarization susceptibility:

$$\chi = \lim_{E \to 0} \frac{\partial}{\partial E} p = \frac{1}{2} \sum_{ij} (2 - i)(2 - j)\, \chi_{ij} \tag{14}$$

In the following we aim at calculating the critical indices β_e and γ_e related to p and χ, respectively. Again, we restrict our investigation to the spin-flip-symmetric model. In order to apply the CAM, we have to determine the mean-field-type behavior of our order parameter. For this

purpose consider the Taylor expansion of the function $\mathscr{F}_L(\{a_i\}, y)$ around the critical point $\{\{a_i^c\}, \bar{y}^c = 1\}$ in powers of $da_i = a_i - a_i^c$ and $dy = y - 1$:

$$\mathscr{F}_L(\{a_i\}, y) = \mathscr{F}_L(\{a_i^c\}, 1) + \sum_i (\partial_{a_i} \mathscr{F}_L)\Big|_c da_i + \frac{1}{2!} \sum_{i,j} (\partial_{a_i a_j} \mathscr{F}_L)\Big|_c da_i\, da_j$$
$$+ \sum_i (\partial_{y a_i} \mathscr{F}_L)\Big|_c da_i\, dy + \frac{1}{2!} \sum_i (\partial_{y y a_i} \mathscr{F}_L)\Big|_c da_i (dy)^2$$
$$+ \frac{1}{3!} \sum_i (\partial_{yyy} \mathscr{F}_L)\Big|_c (dy)^3 + \frac{1}{4!} \sum_i (\partial_{yyyy} \mathscr{F}_L)\Big|_c (dy)^4 + \cdots \quad (15)$$

Here we have omitted the terms with vanishing derivatives $\partial_y \mathscr{F}_L|_c$ and $\partial_{yy} \mathscr{F}_L|_c$ and the ones which will turn out to be of order higher than $(da)^2$. The symbol $|_c$ means that the derivatives should be taken at the critical point. The stationarity condition (5) implies a third-order polymial in $dy = y - 1$,

$$\frac{1}{3!} (\partial_{yyyy} \mathscr{F}_L)\Big|_c (dy)^3 + \frac{1}{2!} (\partial_{yyy} \mathscr{F}_L)\Big|_c (dy)^2$$
$$+ \sum_i (\partial_{y y a_i} \mathscr{F}_L)\Big|_c da_i\, dy + \sum_i (\partial_{y a_i} \mathscr{F}_L)\Big|_c da_i = 0 \quad (16)$$

With regard to the fact that $dy = 0$ always has to be a solution and that it is a threefold degenerate solution at the critical point $da_i = 0$, the derivative $(\partial_{yyy} \mathscr{F}_L)|_c$ and the sum $\sum_i (\partial_{y y a_i} \mathscr{F}_L)|_c\, da_i$ must vanish, which has been verified also numerically. Consequently, besides the trivial solution

$$dy = 0 \quad (17a)$$

reflecting the constancy of the root $y = 1$, there exist two other solutions to (16),

$$dy_{1,2} = \pm \left[- \frac{6 \sum_i (\partial_{y y a_i} \mathscr{F}_L)|_c\, da_i}{(\partial_{yyyy} \mathscr{F}_L)|_c} \right]^{1/2} \quad (17b)$$

which reflect the split between the self-dual roots y_1, y_2 in the low-temperature phase. Taking into account the relations

$$a_i \partial_{a_i} \mathscr{F}_L \sim a_i (\partial_{a_i} \mathscr{F}_L)|_c + a_i (\partial_{y a_i} \mathscr{F}_L)|_c\, dy$$
$$da_i = a_i \log a_i\, dt_L / t_L$$

with $dt_L = t - t_L^c$ and (13), it is easy to show that the polarization exhibits the mean-field-type singular behavior with the classical critical index 1/2,

$$p \sim \pm \bar{p}_L \left(1 - \frac{t_L^c}{t}\right)^{1/2} \tag{18a}$$

where the prefactor is obtained in the form

$$\bar{p}_L = -\frac{1}{2} \sum_i (2-i)\, a_i (\partial_{ya_i} \mathscr{F}_L) \left(- \frac{6 \sum_j (\partial_{yya_j} \mathscr{F}_L)\, a_j \log a_j}{(\partial_{yyyy} \mathscr{F}_L)} \right)^{1/2} \Bigg|_c \tag{18b}$$

As concerns the generalized susceptibilities (12), after some simple algebra they can be written as

$$\begin{aligned} \chi_{ij} = & -\delta_{ij} \beta a_i \left(\partial_{a_j} + \frac{\partial \bar{y}_L}{\partial a_j} \partial_y \right) \mathscr{F}_L \Big|_{y = \bar{y}_L} \\ & - t a_i a_j \left(\partial_{a_i} + \frac{\partial \bar{y}_L}{\partial a_i} \partial_y \right) \left(\partial_{a_j} + \frac{\partial \bar{y}_L}{\partial a_j} \partial_y \right) \mathscr{F}_L \Big|_{y = \bar{y}_L} \end{aligned} \tag{19}$$

The derivatives of $\bar{y}_L$ with respect to a_i can be deduced from its implicit definition $\partial_y \mathscr{F}_L |_{y = \bar{y}_L} = 0$,

$$\frac{\partial \bar{y}_L}{\partial a_i} = - \frac{\partial_{ya_i} \mathscr{F}_L}{\partial_{yy} \mathscr{F}_L} \Big|_{y = \bar{y}_L} \tag{20}$$

Consequently, the singular part of χ_{ij} reads

$$\chi_{ij} \sim t a_i a_j \frac{(\partial_{ya_i} \mathscr{F}_L)(\partial_{ya_j} \mathscr{F}_L)}{\partial_{yy} \mathscr{F}_L} \Big|_{y = \bar{y}_L} \tag{21}$$

Expanding $\partial_{yy} \mathscr{F}_L$ around the critical temperature from the high-temperature side,

$$\partial_{yy} \mathscr{F}_L |_{y = \bar{y}_L} = \sum_i \left[(\partial_{yya_i} \mathscr{F}_L)\, a_i \log a_i \right] \Big|_c \frac{t - t_L^c}{t} \tag{22}$$

we see that χ_{ij} displays the mean-field-type singularity (with the critical exponent equal to 1) as $t \to t_L^{c-}$,

$$\chi_{ij} \sim (\bar{\chi}_{ij})_L \frac{t^2}{t_L^c - t} \tag{23a}$$

$$(\bar{\chi}_{ij})_L = - \frac{a_i a_j (\partial_{ya_i} \mathscr{F}_L)(\partial_{ya_j} \mathscr{F}_L)}{\sum_k \left[(\partial_{yya_k} \mathscr{F}_L)\, a_k \log a_k \right]} \Bigg|_c \tag{23b}$$

Owing to the spin-flip symmetry, only three of the susceptibilities, say χ_{00}, χ_{01}, χ_{11}, are independent: $\chi_{00}=\chi_{44}=-\chi_{04}$, $\chi_{01}=-\chi_{03}=-\chi_{14}=\chi_{34}$, $\chi_{11}=\chi_{33}=-\chi_{13}$ ($\chi_{ij}=0$ as soon as one of its subscripts is equal to 2). Consequently, χ of (14) is given by

$$\chi \sim \bar{\chi}_L \frac{t^2}{t_L^c - t} \tag{24a}$$

$$\bar{\chi}_L = 8(\bar{\chi}_{00})_L + 8(\bar{\chi}_{01})_L + 2(\bar{\chi}_{11})_L \tag{24b}$$

The formulas (18) and (24), revealing the mean-field-type character of the approximation used, enable us to apply the CAM for calculating β_e and γ_e. According to the CAM, the mean-field critical coefficients $\bar{p}_L$ and $\bar{\chi}_L$ diverge as the approximation order L becomes large as follows:

$$\bar{p}_L \sim L^{\delta\beta_e/\nu} \tag{25a}$$

$$\bar{\chi}_L \sim L^{\delta\gamma_e/\nu} \tag{25b}$$

By combining Eqs. (25) and (10), we get

$$\bar{p}_L \sim \frac{1}{(t^c - t_L^c)^{\delta\beta_e}} \tag{26a}$$

$$\bar{\chi}_L \sim \frac{1}{(t^c - t_L^c)^{\delta\gamma_e}} \tag{26b}$$

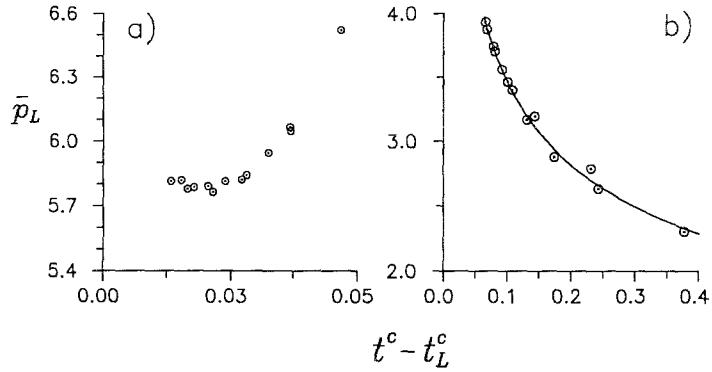

Fig. 4. The polarization coefficient $\bar{p}_L$ vs. $t^c - t_L^c$ for two particular values of the vertex energy ε_1, namely (a) $\varepsilon_1 = 3$, reflecting the evident convergence of data to a finite value ($\delta\beta_e = 0$) in the immediate neighborhood of the B8V critical point, and (b) the Ising-like energy $\varepsilon_1 = 0.6275...$.

Exponents $-\delta\beta_e$ and $\delta\gamma_e$ clearly represent the respective corrections to the classical exponents 1/2 and 1,

$$\beta_e = \tfrac{1}{2} - \delta\beta_e \tag{27a}$$

$$\gamma_e = 1 + \delta\gamma_e \tag{27b}$$

We have calculated the mean-field critical coefficients $\bar{p}_L$ and $\bar{\chi}_L$ along the critical line for each order L. First of all let us convince ourselves that our series possesses the property of coherent anomaly. Figures 4–6 show that

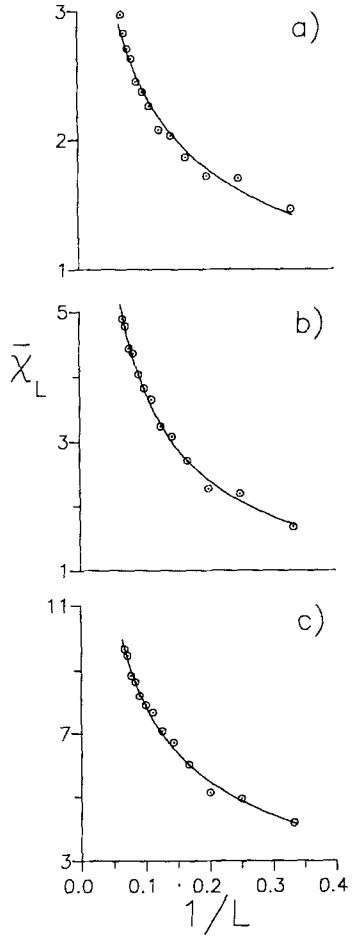

Fig. 5. The coherent-anomaly coefficient $\bar{\chi}_L$ vs. $1/L$ calculated for (a) $\varepsilon_1 = 2$, (b) $\varepsilon_1 = 0.6275...$, (c) $\varepsilon_1 = 0.1$. Data are fitted according to the large-L asymptotics (25b).

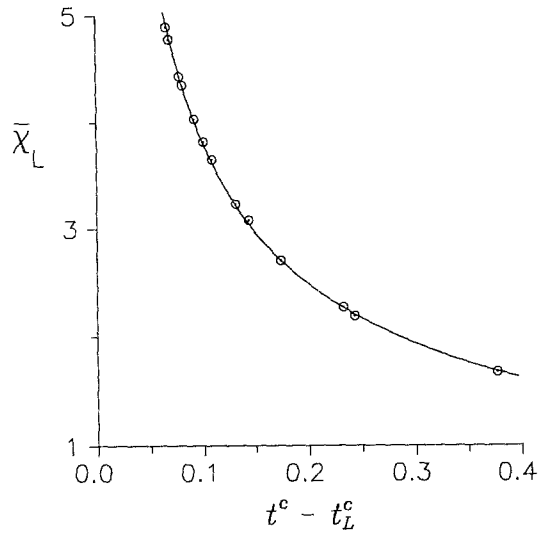

Fig. 6. The dependence of the coefficient $\bar{\chi}_L$ on $t^c - t_L^c$ for the Ising-like vertex energy $\varepsilon_1 = 0.6275...$: the data exhibit perfectly the expected power-law behavior (26b).

our data obey the power-low behavior (25), (26) even in the lower approximations. Naturally, one can observe a slight dispersion of data, but the coherent anomaly scaling is undoubted.

Again we have a possibility to check the accuracy of our results by confrontation with the B8V case ($\varepsilon_1 \to \infty$), for which the exact value of $\beta_e = 1/2$ is known.[2] Figure 4a reports the $\bar{p}_L$ data for $\varepsilon_1 = 3$, reflecting the typical plot in the immediate neighborhood of the B8V critical point. It is seen that the data evidently converge to a finite value. This corresponds to zero correction $\delta\beta_e$ to the mean-field critical index and results in $\beta_e = 1/2$, in a full agreement with the exact value. The boundary between this convergence and the CAM divergence is hardly detectable from a finite number of $\bar{p}_L$ data. That is why in the plot of β_e vs. a_1^c in Fig. 3 the results near the B8V critical point ($a_1^c = 0$), depicted by a changed dashing of the line, are only interpolated. The dependence of γ_e on a_1^c is presented in the same figure. Unfortunately, no exact results are available for this index. We conclude that the behavior of the critical indices β_e and γ_e corroborates universality breaking.

5. CONCLUDING REMARKS

We have proposed the variational series expansion method for the investigation of the phase diagram and critical behavior of the symmetric

16-vertex model. This method seems to be efficient in an approximate description of vertex models partially because it provides meaningful results even in the lowest orders. The power of the method consists in the fact that it is based on an internally consistent series expansion which is adaptable in the low- as well as high-temperature phases and, moreover, possesses the property of coherent anomaly scaling. It stands to reason that this approach is not restricted to the symmetric vertex models.

We want to give one remark concerning the accuracy of the obtained critical indices. In general it is difficult to estimate the errors in the CAM. Therefore, the best way to get a feeling about errors is the comparison with the known exact results. As we have seen, for the critical index ν the absolute differences did not exceed 0.1. Unfortunately, apart from the B8V critical point, no exact information about the exponents β_e and γ_e is available. We believe that the accuracy of their determination is comparable with that attained for ν. This is supported by the fact that the estimates of β_e and γ_e deduced from relations (25) and (26) are practically identical.

One interesting fact is to be noted. The plots of the calculated indices clearly show that Suzuki's idea of weak universality does not apply to the polarization-related exponents β_e, γ_e in the form as formulated for the magnetic indices β, γ.

As far as β and γ are concerned, it is in principle possible to calculate them using the present method restricted to the manifold on which the mapping onto the Ising model can be performed (see Fig. 2). However, the formulas to be handled are quite complicated and we are not able to define the Ising magnetization unambiguously without restriction to the Ising subspace.

Various interesting problems remain to be solved in connection with the model under discussion, e.g., the calculation of other critical indices, checking the scaling relations, or determining the conformal charge of the model.

In conclusion, the present work shows that nonuniversal behavior may not be as exceptional as is generally believed and raises the interesting question of whether the universality is restored in three dimensions for this class of models.

APPENDIX A

In this appendix we explain how we determined the coefficients $c(n_1, n_2, n_3, n_4)$ of the series expansion (4), (4′) up to the $L = 15$ degree. In order to avoid the enumeration of a plethora of all graphs embedded in the square lattice we proceeded in two steps. In the first step, we generated the "no-free-ends" (NFE) expansion considering only the graphs without

vertices having a single neighbor. Its treatment is a much simpler task than the direct computation of the general expansion, because the NFE graphs represent only a small fraction of all possible graphs. From the NFE expansion we calculated the general series expansion for the S16V by using an *algebraic* procedure based on the gauge transformation. We now describe the method in detail.

First we introduce an auxiliary function $y^* = y^*(\{a_i\})$ defined by the equation

$$\tilde{a}_1(y^*(\{a_i\})) = 0 \tag{A.1}$$

which means that the gauge transformation (2) with the parameter $y = y^*$ transforms the a_i in such a way that the resulting weight $\tilde{a}_1$ equals to zero. From (2) one can derive the following series for the function y^*:

$$y^*(\{a_i\}) = a_1/a_0 + 3a_1a_2/a_0^2 + \cdots = \sum_n y_n^* \tag{A.2a}$$

where y_n^* are of the order n in a_i/a_0 and can be calculated recursively as follows:

$$y_n^* = \frac{a_1}{a_0}\delta_{n,1} + \frac{3a_2}{a_0}y_{n-1}^* + \frac{3(a_3-a_1)}{a_0}\sum_{\substack{i,j \\ i+j=n-1}} y_i^* y_j^* + \frac{a_4-3a_2}{a_0} \times \sum_{\substack{i,j,k \\ i+j+k=n-1}} y_i^* y_j^* y_k^* - \frac{a_3}{a_0}\sum_{\substack{i,j,k,l \\ i+j+k+l=n-1}} y_i^* y_j^* y_k^* y_l^* \tag{A.2b}$$

The above-mentioned NFE series is obtained by inserting $y = y^*$ into the formal series expansion of the form (4), (4′):

$$-\mathcal{F}(\{a_i\}, y^*) = \log[\tilde{a}_0(y^*)] + \sum_{jkl} n_{jkl}\left(\frac{\tilde{a}_2(y^*(\{a_i\}))}{\tilde{a}_0(y^*(\{a_i\}))}\right)^j \left(\frac{\tilde{a}_3(y^*(\{a_i\}))}{\tilde{a}_0(y^*(\{a_i\}))}\right)^k \times \left(\frac{\tilde{a}_4(y^*(\{a_i\}))}{\tilde{a}_0(y^*(\{a_i\}))}\right)^l \tag{A.3}$$

with $n_{jkl} = c(0, j, k, l)$. Although this expansion contains only the contributions of the NFE graphs, it carries the full information about the original model. It is a relatively simple task to calculate the numbers n_{jkl}. We first enumerated and stored all connected NFE graphs and then we calculated the contributions of graphs consisting of two and three disconnected components (the lowest contribution of the NFE graphs with four components,

Table I. Complete List of the Contributions of the One-, Two-, and Three-Component NFE Graphs for Constructing Series Expansion (4) up to $L=15$ Order

One-component contributions

j	k	l	n_{jkl}	j	k	l	n_{jkl}	j	k	l	n_{jkl}	j	k	l	n_{jkl}
4	0	0	1	6	2	1	8	7	8	0	1148	10	0	0	28
4	2	0	2	6	2	2	8	8	0	0	7	10	0	1	40
4	4	0	2	6	4	0	62	8	0	1	8	10	0	2	36
4	4	1	1	6	4	1	48	8	0	2	6	10	0	3	22
4	6	0	2	6	4	2	32	8	0	4	1	10	0	4	4
4	6	2	2	6	4	3	28	8	2	0	162	10	0	5	8
4	8	0	4	6	4	4	6	8	2	1	152	10	2	0	1276
4	8	3	2	6	6	0	146	8	2	2	102	10	2	1	1660
4	10	0	2	6	6	1	128	8	2	3	56	10	2	2	1346
4	10	1	4	6	6	2	188	8	2	4	24	10	2	3	944
5	2	1	4	6	6	3	96	8	2	5	16	10	4	0	11888
5	4	0	4	6	8	0	268	8	4	0	940	10	4	1	19474
5	4	1	8	6	8	1	346	8	4	1	1200	11	2	0	2864
5	4	2	8	7	2	0	26	8	4	2	1112	11	2	1	4004
5	6	0	8	7	2	1	32	8	4	3	832	11	2	2	3834
5	6	1	24	7	2	2	20	8	6	0	3604	11	4	0	34596
5	6	2	8	7	2	3	12	8	6	1	5580	12	0	0	124
5	6	3	16	7	2	4	4	9	2	0	312	12	0	1	208
5	6	4	4	7	4	0	140	9	2	1	376	12	0	2	220
5	8	0	8	7	4	1	252	9	2	2	310	12	0	3	184
5	8	1	48	7	4	2	224	9	2	3	212	12	2	0	9888
5	8	2	40	7	4	3	136	9	2	4	124	12	2	1	15452
5	10	0	30	7	4	4	80	9	4	0	2548	13	2	0	23704
6	0	0	2	7	6	0	420	9	4	1	4672	14	0	0	588
6	0	1	2	7	6	1	1192	9	4	2	4808	14	0	1	1120
6	2	0	20	7	6	2	1058	9	6	0	12056				

Two-component contributions

j	k	l	$-n_{jkl}$	j	k	l	$-n_{jkl}$	j	k	l	$-n_{jkl}$	j	k	l	$-n_{jkl}$
8	0	0	9/2	9	4	1	296	10	4	0	1980	12	0	1	208
8	2	0	24	9	4	2	152	10	4	1	1400	12	0	2	150
8	4	0	61	9	6	0	312	11	2	0	448	12	2	0	4052
8	4	1	16	10	0	0	24	11	2	1	728	12	2	1	4592
8	6	0	112	10	0	1	28	11	2	2	564	13	2	0	7552
8	6	1	40	10	2	0	366	11	4	0	4076	14	0	0	784
9	2	1	60	10	2	1	208	12	0	0	137	14	0	1	1446
9	4	0	64	10	2	2	144								

Three-component contributions

j	k	l	n_{jkl}	j	k	l	n_{jkl}	j	k	l	n_{jkl}	j	k	l	n_{jkl}
12	0	0	194/6	12	2	0	290	14	0	0	290	14	0	1	362

namely squares, is of the 16th order). Let us note that the CPU time required for this job did not exceed 15 min on the HP Apollo 9000/720 workstation.

Having determined n_{jkl} up to the $L=15$ order, we expanded (A.3) in powers of a_i/a_0 up to the same order with the help of (A.2) and the transformation formula for $\tilde{a}_0$, (2). In such a way we obtained the graphical series expansion of the free energy of a general S16V:

$$-\mathscr{F}(\{a_i\}, y=0) = \log(a_0) + \sum_{n_1,n_2,n_3,n_4} c(n_1, n_2, n_3, n_4) \times \left(\frac{a_1}{a_0}\right)^{n_1} \left(\frac{a_2}{a_0}\right)^{n_2} \left(\frac{a_3}{a_0}\right)^{n_3} \left(\frac{a_4}{a_0}\right)^{n_4} \tag{A.4}$$

The computer time needed for these algebraic manipulations was negligible in comparison with that needed for the enumeration of the NFE graphs. The coefficients of the polynomials $\mathscr{F}^{(n)}$ in (4′) do not depend on the particular choice of the parameter y. This is why $c(n_1, n_2, n_3, n_4)$ in (A.4) and in (4′) are identical.

Because the complete list of nonzero $c(n_1, n_2, n_3, n_4)$ contains almost 1000 items, it is impossible to present it in this paper. In Table I we report the data for the NFE expansion (A.3) which enable the reader to calculate $c(n_1, n_2, n_3, n_4)$ straightforwardly by applying the outlined algebraic procedure.

APPENDIX B

In order to explain the root structure of the stationary condition (5) we concentrate on the analytical treatment of the lowest-order variational series expansion

$$-\mathscr{F}_0(\{a_i\}, y) = \log[\tilde{a}_0(y)] \tag{B.1}$$

With regard to the equality $\partial_y \tilde{a}_0 = -4\tilde{a}_1/(1+y^2)$, the stationarity condition (5) is identified with

$$\tilde{a}_1(y) = 0 \tag{B.2}$$

The set of roots $S = \{y_i\}$ to the fourth-degree polynomial (B.2) determines the free energy through the relation

$$-f_0 = \max_{y_i \in S} [\log(\tilde{a}_0(y_i))] \tag{B.3}$$

In the case of the Bethe lattice, owing to the absence of cycles, the excitations from the sea of (+) edge states do not contribute to the partition

function under the condition (B.2). That is why Eqs. (B.2), (B.3) represent the exact solution of the S16V formulated on such a structure, and we will identify them as the Bethe approximation.

From the point of view of the critical properties, the relevant subspace in the vertex-weight parameter space is the self-dual manifold (3). It turns out that the stationarity condition (B.2) can be solved exactly as soon as we restrict the weights $\{a_i\}$ to it. The crucial simplification consists in the fact that one can nullify simultaneously, using the gauge transformation, all vertex weights with an odd number of incident edges in the $(-)$ state, i.e., for each point of the self-dual subspace there exists at least one particular gauge parameter which implies besides the Bethe condition $\tilde{a}_1 = 0$ also $\tilde{a}_3 = 0$. Since

$$\tilde{a}_1 + \tilde{a}_3 = \frac{1}{1+y^2}\left[(y^2-1)(a_1+a_3) + y(a_0-a_4)\right] \tag{B.4}$$

the required gauge parameters, one positive and one negative, read

$$y_+ = \frac{a_4 - a_0 + \left[(a_4-a_0)^2 + 4(a_1+a_3)^2\right]^{1/2}}{2(a_1+a_3)} \tag{B.5a}$$

$$y_- = \frac{a_4 - a_0 - \left[(a_4-a_0)^2 + 4(a_1+a_3)^2\right]^{1/2}}{2(a_1+a_3)} \tag{B.5b}$$

Inserting $y_\pm$ into $\tilde{a}_1 = 0$ and $\tilde{a}_3 = 0$, one can verify that (B.5) are actually solutions to (B.2) as soon as the model lies on the self-dual manifold (3).

The remaining two roots to the Bethe condition (B.2) on the self-dual manifold (3) are readily obtained in the form

$$y_{1,2} = \frac{a_0 a_3 + a_1 a_4 - 3a_2(a_1+a_3) \pm \sqrt{D}}{2a_3(a_1+a_3)} \tag{B.6a}$$

$$D = \left[a_0 a_3 + a_1 a_4 - 3a_2(a_1+a_3)\right]^2 - 4a_1 a_3 (a_1+a_3)^2 \tag{B.6b}$$

and do not imply, except for the critical point, $\tilde{a}_3 = 0$. With respect to the equalities

$$\tilde{a}_0(y_1) = \tilde{a}_0(y_2), \quad \tilde{a}_2(y_1) = \tilde{a}_2(y_2), \quad \tilde{a}_3(y_1) = -\tilde{a}_3(y_2), \quad \tilde{a}_4(y_1) = \tilde{a}_4(y_2) \tag{B.7}$$

these roots reflect the self-dual property of the free energy under the gauge transformation on the manifold (3), $f(y_1) = f(y_2)$ (note that the free energy is invariant under the simultaneous negation of $\tilde{a}_1, \tilde{a}_3$) and we realize that y_1 and y_2 are the above-mentioned self-dual roots.

The interplay among the roots on the self-dual manifold is the following. When the roots y_1 and y_2 are the real positive numbers, i.e., when

$$D \geqslant 0 \tag{B.8a}$$

$$a_0 a_3 + a_1 a_4 - 3a_2(a_1 + a_3) > 0 \tag{B.8b}$$

they are "dominant," i.e., y_1 and y_2 represent the limiting value of the physical solution $\bar{y}$ when approaching the self-dual manifold in the direction of the increasing and decreasing temperature. Because of the self-duality of the free energy $f^{(0)}(y_1) = f^{(0)}(y_2)$, the first derivative of $f^{(0)}$ with respect to a_i is discontinuous and the system undergoes a first-order phase transition.

When

$$D = 0 \tag{B.9a}$$

$$a_0 a_3 + a_1 a_4 - 3a_2(a_1 + a_3) \geqslant 0 \tag{B.9b}$$

the positive roots y_1, y_2 are automatically identical to y_+,

$$y_1 = y_2 = y_+ \tag{B.10}$$

and a second-order phase transition takes place. It is evident that the critical conditions (8) and (B.9a), (B.9b) coincide with one another.

If

$$D < 0 \tag{B.11a}$$

or

$$a_0 a_3 + a_1 a_4 - 3a_2(a_1 + a_3) < 0 \tag{B.11b}$$

the physical solution is the root y_+ and the free energy is an analytical function of the vertex weights a_i. Since the inequality (B.11a) is fulfilled in the limit of infinite temperature ($a_i = 1$ for all i), the corresponding region of the self-dual manifold is connected with the high-temperature disordered regime of the vertex model. The negative root y_- is irrelevant and favors the ground state with all edges in the state $(-)$.

Finally, let us calculate the polarization in the Bethe approximation explicitly, in order to show that the $\bar{y}$ is in relation with this order parameter. We insert an auxiliary two-coordinated vertex into the lattice edge on which we are going to calculate its polarization. This new vertex has the diagonal weight matrix $\mathrm{diag}\{1, -1\}$. Then the polarization is expressed as the ratio of the modified statistical sum over the original

partition function. On the Bethe lattice there are no excitations with a finite number of $(-)$ states even with one new vertex as soon as the condition (B.2) is fulfilled. This is why we need to consider only the ground state which projects out the $(++)$ element from the (transformed) additional vertex weight matrix. In such a way we arrive at the relation between the polarization and the physical solution to the stationarity condition: $p = (1-\bar{y}^2)/(1+\bar{y}^2)$. Let us stress that this simple relation is valid only in the lowest-order approximation.

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